The Influence of Noise on Fractals

Andreas Hamm

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

The weak-noise asymptotics of the blurring effect of noise on fractals can be described by scaling laws. It does not only depend on the geometric properties of the fractals but also on their generating dynamics. This is illustrated with the example of the Feigenbaum attractor.

Fractals are objects with structural details on arbitrarily small length scales. The blurring effect of noise destroys at least the smallest of these details. Going beyond this trivial general observation one can look for an asymptotic relation between the noise strength and the finest resolvable structures ("noise scaling").

This question was probably first posed in connection with the noisy period doubling scenario ([1–3] and references therein) in which the fractal object is the Feigenbaum attractor. Two recent articles dealing with other fractals are [4, 5].

An appropriate tool for deriving asymptotic results of the type discussed in this paper is the so called quasipotential method (e.g. [2, 6]). However, for a sketchy understanding of these results without following the full derivation [2, 7] the heuristic arguments which I am describing below may be helpful.

When studying the influence of noise on a fractal it is not enough to specify the fractal as a geometric object – it is rather the dynamical mechanism generating the fractal, which is affected by the noise. This point can be studied with the Feigenbaum attractor (see [8]). In the original context this object is the attractor of a one-dimensional map F satisfying

$$F(x) = -\alpha F(F(\alpha^{-1} x)), \tag{1}$$

where $\alpha = 2.503\ldots$, and having a quadratic maximum $F(0) = 1$ at 0. Defining $x_k = F^k(0)$ and $\Delta_i^{(n)} = [\min(x_i, x_{2^n+i}), \max(x_i, x_{2^n+i})]$ the Feigenbaum attractor is $A = \bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} \Delta_i^{(n)}$.

But there are other ways to generate A. It can be interpreted as the repeller of the expanding map

$$G(x) = \begin{cases} -\alpha x & x \in \Delta_2^{(1)}, \\ -\alpha F(x) & x \in \Delta_1^{(1)}, \end{cases} \tag{2}$$

or as the attractor of the contracting random map

$$H(x) = -\frac{1}{\alpha}[\gamma x + (1-\gamma) F|_{\Delta_1^{(1)}}^{-1}(x)], \tag{3}$$

where γ is a random variable taking on the values 0 and 1 with equal probability.

Let E be any of the maps F, G, or H and consider a sequence $v_f := (v_1, v_2, \ldots, v_f)$ of indices such that $x_{v_{i+1}} = E(x_{v_i})$ for $1 \le i < f$, numerating part of an orbit running along the boundary of A.

When white noise is added to the deterministic map E, a point x is no longer mapped to $E(x)$ but to $E(x) + \xi$, where ξ is a random variable which is assumed here to have a probability density of the asymptotic form

$$p_\xi^{(\sigma, r)}(\xi') \asymp \exp\left[-\frac{1}{r}\left|\frac{\xi'}{\sigma}\right|^r\right]. \tag{4}$$

Here, r is a parameter controlling the shape of the noise distribution (for Gaussian noise one has $r = 2$; for $r \to \infty$ one approaches the situation of localised noise), σ measures the noise strength, and $S(\sigma) \asymp T(\sigma)$ is a shorthand notation for $\lim_{\sigma \to 0} \ln S(\sigma)/\ln T(\sigma) = 1$.

The noise changes the sequence (x_{v_i}) into a sequence of random variables, (y_{v_i}). After some calculation one obtains in small-noise approximation

$$y_{v_i} = x_{v_i} + \zeta_i, \tag{5}$$

where the random variable ζ_i has asymptotically a probability density

$$p_{\zeta_i}(\xi') \asymp \exp\left[-\frac{1}{r}C_{\nu_i}^{(r)}\left|\frac{\xi'}{\sigma}\right|^r\right] \qquad (6)$$

depending on σ, r, and ν_i. The coefficients

$$C_{\nu_f}^{(r)} = C_{\nu_1}^{(r)}\left[\sum_{j=1}^{f}\prod_{k=j}^{f-1}|E'(x_{\nu_k})|^{r/(r-1)}\right]^{1-r} \qquad (7)$$

$$= C_{\nu_i}^{(r)}|(E^{f-1})'(x_{\nu_1})|^{-r}\left[\sum_{j=1}^{f}|(E^{j-1})'(x_{\nu_1})|^{-r/(r-1)}\right]^{1-r}$$

will turn out to be decisive for the noise scaling.

Let Λ_{ν_f} be the gap of A which has x_{ν_f} as a boundary point. In the presence of noise, this gap can be resolved as long as, within this gap, p_{ζ_f} stays smaller than a certain threshold of resolution which is determined by the method used for observing the system. Thus we see from (6) that the critical noise strength for destruction of Λ_{ν_f} shows the behaviour

$$\sigma_{\nu_f}^* \sim \text{const}\,(C_{\nu_f}^{(r)})^{1/r}|\Lambda_{\nu_f}|. \qquad (8)$$

This means that the critical noise strength is not only related to the length $|\Lambda_{\nu_f}|$ of the gap, as one would expect naively, but also to the coefficients (7) – and these depend strongly on which of the maps F, G, or H is inserted for E.

1. $E = G$: Since G is expanding (i.e., there is a constant $\beta > 1$ such that $|G'(x)| \geq \beta$ everywhere) one can see from (8) and (7) that there are positive constants m, M such that

$$m \leq \sigma_{\nu_f}^*/\sigma_{\nu_1}^* \leq M. \qquad (9)$$

But since $\sigma_{\nu_1}^*$ can be chosen arbitrarily small (by starting with a sufficiently small gap Λ_{ν_1}) this shows that $\sigma_{\nu_f}^*$ is arbitrarily small for all gaps. Not surprisingly, an arbitrarily small noise strength is enough to blur all gaps in A if it is interpreted as a repeller.

2. $E = H$: Since H is contracting (i.e., there is a constant $\bar{\beta} < 1$ such that $|H'(x)| \leq \bar{\beta}$ everywhere) one can see from (7) that there are positive constants n, N such that

$$n \leq C_{\nu_f}^{(r)}/C_{\nu_1}^{(r)} \leq N. \qquad (10)$$

Together with (8) this means that

$$\sigma_{\nu_f}^* \asymp |\Lambda_{\nu_f}|; \qquad (11)$$

the scaling behaviour of the critical noise strength is trivially related to the length scaling of the gaps.

3. $E = F$: In this case the derivative of F can take on values both larger as well as smaller than 1 in modulus. Therefore, the scaling behaviour of the coefficients (7) is more interesting here. Because of $\Delta_j^{(n)} = F^{j-1}(\Delta_1^{(n)})$, one obtains from (7)

$$C_{(1,2,\ldots,2^n)}^{(r)} \approx C_{(1)}^{(r)}|\Delta_{2^n}^{(n)}|^{-r}\left[\mathcal{Z}_n\left(-\frac{r}{r-1}\right)\right]^{1-r}, \qquad (12)$$

where

$$\mathcal{Z}_n(\beta) := \sum_{j=1}^{2^n}|\Delta_j^{(n)}|^{\beta} \qquad (13)$$

is the partition function known from the Thermodynamic Formalism (see e.g. [9]). The free energy $\beta\mathcal{F}(\beta)$ describes the scaling of the partition function: $\mathcal{Z}_n(\beta) \asymp \exp(-\beta\mathcal{F}(\beta)n)$. Thus, (8) and (12) lead to

$$\sigma_{2^n}^* \asymp \exp\left(-\mathcal{F}\left(-\frac{r}{r-1}\right)n\right). \qquad (14)$$

This means that $\varkappa_r := \exp\left(\mathcal{F}\left(-\frac{r}{r-1}\right)\right)$ is the noise scaling factor by which the noise strength has to be reduced in order to resolve one more generation of gaps in A. Its dependence on r shows that (in contrast to the cases $E = G$ and $E = H$) the shape of the noise distribution is essential for the noise scaling in the Feigenbaum scenario, a fact that has been overlooked for many years by overemphasising the role of $\varkappa_2 \approx 6.62$. The noise scaling factor for localised noise can be obtained by taking the limit $r \to \infty$: $\varkappa_\infty = \exp(\mathcal{F}(-1)) \approx 8.49$. A direct proof without the detour via finite r can be given by developing arguments from [8, 10].

The example of the fractal A has shown that noise scaling in fractals depends crucially on the nature of the generating dynamics. If it is hyperbolic, arbitrarily weak noise is enough to blur all structures in unstable directions, whereas the structures in stable directions which are – roughly speaking – larger than the standard deviation of the noise can still be observed (see [11] for an example). For non-hyperbolic dynamics the relation between length scales and critical noise strengths may be much more indirect, as the case $E = F$ has demonstrated.

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